

Hamiltonian Extension and Eigenfunctions for a Time Dispersive Dissipative String

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ABSTRACT. We carry out a detailed analysis of a time dispersive and dissipative (TDD) string, using our recently developed theories of conservative and Hamiltonian extensions of TDD systems. This analysis of the TDD string includes, in particular: (i) an explicit construction of its conservative Hamiltonian extension, consisting of the original string coupled to “hidden strings”; (ii) explicit formulas for energy and momentum densities in the extended system, providing a transparent physical picture accounting precisely for the dispersion and dissipation; (iii) the eigenmodes for the extended string system, which provide an eigenmode expansion for solutions to the TDD wave equation governing the solution to the TDD string. The obtained results provide a solid basis for the rigorous treatment of the long standing problem of scattering by a TDD scatterer, illustrated here by the computation of scattering states for a string with dissipation restricted to a half line.

1. Introduction

The need for a Hamiltonian description of dissipative systems has long been known, having been emphasized by Morse and Feshbach [5, Chapter 3.2] forty years ago. Recently we have introduced conservative [1] and then Hamiltonian [2, 3] theories of time dispersive and dissipative (TDD) systems addressing that need. The extended Hamiltonian is constructed by coupling the given TDD system to a system of “hidden strings” in a canonical way so that it has a transparent interpretation as the system energy. It turns out that such a system of hidden strings is, in fact, a canonical heat bath as described in [4, Section 2; 6, Section 2].

This work is intended as a self contained companion to the papers [1–3], illustrating the phenomena described there through explicit calculations for the typical example of a TDD string.

To keep everything elementary, let us consider a scalar wave equation in one spatial dimension, such as might be used to describe wave propagation along a

2000 *Mathematics Subject Classification.* Primary 37L50; Secondary 35P10.

The effort of the first author was sponsored by the Air Force Office of Scientific Research, Air Force Materials Command, USAF, under grant number FA9550-04-1-0359.

The second author received travel and materials support under the aforementioned USAF grant and was supported by a National Science Foundation post-doctoral fellowship.

This is the final form of the paper.

homogeneous string. In the absence of dispersion, the displacement $\phi(x, t)$ of the string at position $x \in \mathbb{R}$ and time $t \in \mathbb{R}$ evolves by the 1D wave equation

$$(1.1) \quad \partial_t^2 \phi(x, t) - \gamma \partial_x^2 \phi(x, t) = f(x, t), \quad x, t \in \mathbb{R},$$

where

- γ is the tension of the string.
- we have taken units in which the mass per unit length is 1.
- $f(x, t)$ is an external *driving force* per unit length, usually supposed bounded and compactly supported.

For the most part, we consider (1.1) with the string *at rest* at $t = -\infty$,

$$(1.2) \quad \lim_{t \rightarrow -\infty} \phi(x, t) = \lim_{t \rightarrow \infty} \partial_t \phi(x, t) = 0.$$

Thus the solution $\phi(x, t)$ is a function of the driving force $f(x, t)$, indeed

$$(1.3) \quad \phi(x, t) = \frac{1}{2v} \int_{-\infty}^t \int_{x-v(t-t')}^{x+v(t-t')} f(y, t') \, dy \, dt'$$

with $v = 1/\sqrt{\gamma}$ the speed of propagation on the string.

In this paper we consider a modification of (1.1) incorporating friction in the form of a dissipative term with time dispersion. Specifically, we consider the following equation

$$(1.4) \quad \partial_t \left\{ \partial_t \phi(x, t) + \int_{-\infty}^t \chi(x, t-t') \partial_{t'} \phi(x, t') \, dt' \right\} - \gamma \partial_x^2 \phi(x, t) = f(x, t),$$

with $\chi(x, \tau)$ a given function, called the susceptibility, satisfying a power dissipation condition—(1.23) below. The physical idea behind (1.4) is as follows. The wave equation (1.1) can be expressed in terms of the string momentum $\pi(x, t) = \partial_t \phi(x, t)$ as

$$(1.5) \quad \partial_t \pi(x, t) = \text{sum of all forces at } x = \underbrace{f(x, t)}_{\text{driving force}} + \underbrace{\gamma \partial_x^2 \phi(x, t)}_{\text{string tensile force}}.$$

In the damped string (1.4), the basic relation $\partial_t \pi =$ “sum of all forces” still holds, but the simple relationship $\pi = \partial_t \phi$ between the string momentum and velocity is replaced by the *material relation*

$$(1.6) \quad \pi(x, t) = \partial_t \phi(x, t) + \int_{-\infty}^t \chi(x, t-t') \partial_{t'} \phi(x, t') \, dt'.$$

Equation (1.6) is supposed to be a phenomenological relation describing the interaction of the string with a surrounding medium, expressing the fact that some of the string momentum is absorbed by the medium and then partially retransmitted to the string with delay.

The wave equation (1.1) may be expressed as a Hamiltonian system. As described in [2, 3], the procedure of going from (1.1) to (1.4) is naturally understood in this context and indeed (1.4) *can be derived from a larger Hamiltonian system, with additional variables*. The additional variables of this extension may be interpreted as describing a “hidden string,” with internal coordinate s , attached to each point x of the physical string (see Figure 1). These “hidden strings” are coupled to the physical string via a coupling function $\zeta(x, s)$, but do not interact directly with

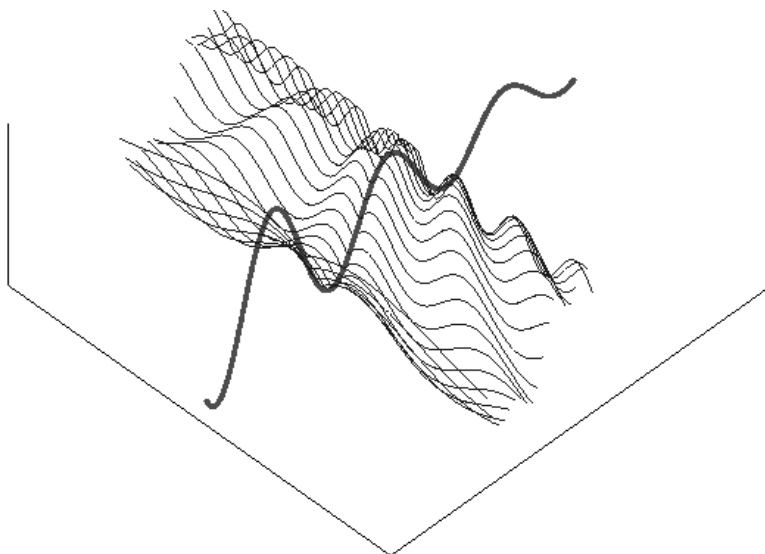


FIGURE 1. The extended system consists of the physical string (thick) coupled to an independent “hidden string” (thin) at every point of dissipation. This Hamiltonian system models exactly the dissipation and dispersion in the physical string via the exchange of energy with the hidden strings.

one another.¹ Upon “integrating out” the hidden strings, the result is an effective dynamics of the form (1.4) for the physical string, with susceptibility

$$(1.7) \quad \chi(x, \tau) = \frac{1}{2} \int_{\mathbb{R}} \zeta(s, x) \int_{s-\tau}^{s+\tau} \zeta(x, r) dr ds.$$

Different choices of the coupling function $\zeta(x, s)$ produce different susceptibilities $\chi(x, \tau)$, and furthermore [2, 3] any susceptibility satisfying the power dissipation condition — (1.23) below — can be obtained in this way.

Given the role of the TDD wave equation (1.4) as a phenomenological description for the interaction of the string with its environment, it is not surprising that an extended system exists which produces the effective dynamics (1.4), at least approximately. However, the main point of [2, 3], which we explain below, is that it is possible to construct explicitly an extended system which *exactly* reproduces (1.4). Further, using that system we can derive simple expressions for quantities like the energy and wave momentum densities for the damped string in terms of the susceptibility χ .

In addition, as illustrated below, the extended system clarifies the nature and role of eigenfunctions in a TDD system by providing a natural eigenmode expansion

¹The lack of interaction between hidden strings is not essential. Indeed the methods of [1–3] allow to consider systems with “spatial dispersion” in the material relation

$$\pi(x, t) = \partial_t \phi(x, t) + \int_{\mathbb{R}} \int_0^{\infty} \chi(x', \tau) \partial_t \phi(x - x', t - \tau) d\tau dx'.$$

The extensions for such systems involve interaction between the hidden strings, but for simplicity we do not pursue this here.

for solutions to (1.4). In this context, an eigenfunction $\exp(-i\omega t)\phi_\omega(x)$ is a time periodic steady state solution to (1.4). For the simple nondissipative string (1.1), an eigenmode solves

$$(1.8) \quad \gamma \partial_x^2 \phi_\omega(x) = -\omega^2 \phi_\omega(x),$$

which is to say it is a linear superposition of the plane waves $\exp(\pm i\omega x/\sqrt{\gamma})$. For the TDD string (1.4), the dissipation induced by the dispersion in (1.6) may preclude the existence of a steady state solution without a source term. However, the extended system is without dissipation, and it is no trouble to construct its eigenmodes. We find below that the resulting eigenmode equation for ϕ_ω

$$(1.9) \quad \gamma \partial_x^2 \phi_\omega(x) = -\omega^2 (1 + \operatorname{Re} \widehat{\chi}(x, \omega)) \phi_\omega(x) - i\omega^2 \operatorname{Im} \widehat{\chi}(x, \omega) g_\omega(x),$$

has a source term involving an arbitrary function g_ω , which depends on the configuration of the hidden strings. Here $\widehat{\chi}$ is the τ -Fourier transform of χ ,

$$(1.10) \quad \widehat{\chi}(x, \omega) = \int_0^\infty e^{i\omega\tau} \chi(x, \tau) d\tau.$$

Let us now sketch the construction of the Hamiltonian extension. To start we write (1.1) as a first order system, introducing the momentum density $\pi(x, t) = \partial_t \phi(x, t)$, so that

$$(1.11) \quad \begin{aligned} \partial_t \phi(x, t) &= \pi(x, t) \\ \partial_t \pi(x, t) &= \gamma \partial_x^2 \phi(x, t) + f(x, t). \end{aligned}$$

This first order system is Hamiltonian, with symplectic form

$$(1.12) \quad J(\phi_1, \pi_1; \phi_2, \pi_2) = \int_{\mathbb{R}} (\phi_1(x) \pi_2(x) - \phi_2(x) \pi_1(x)) dx$$

and time dependent Hamilton function

$$(1.13) \quad H_f(\phi, \pi, t) = \frac{1}{2} \int_{\mathbb{R}} (\pi(x)^2 + \gamma |\partial_x \phi(x)|^2) dx - \int_{\mathbb{R}} f(x, t) \phi(x) dx.$$

As the system is nonautonomous, $H_f(\pi(\cdot, t), \phi(\cdot, t), t)$ is not conserved. Instead along a trajectory to (1.11), we have

$$(1.14) \quad \frac{dH_f}{dt} = \partial_t H_f(\phi(\cdot, t), \pi(\cdot, t), t) = - \int_{\mathbb{R}} \partial_t f(x, t) \phi(x, t) dx.$$

We take the internal energy of the (undamped) string to be the Hamilton function $H_0(\phi, \pi)$ of the system with $f \equiv 0$. Thus, using (1.14) we obtain

$$(1.15) \quad \frac{dH_0}{dt} = \frac{dH_f}{dt} + \frac{d}{dt} \int_{\mathbb{R}} f(x, t) \phi(x, t) dx = \int_{\mathbb{R}} f(x, t) \partial_t \phi(x, t) dx.$$

In other words, the rate of work done on the system is the integral of the external force times the velocity — “power = force \times velocity.”

Observe that the Hamiltonian H_0 can be written

$$(1.16) \quad H_0(\phi, \pi) = \frac{1}{2} (\|K_\pi \pi\|_{L^2(\mathbb{R})}^2 + \|K_\phi \phi\|_{L^2(\mathbb{R})}^2)$$

where

$$(1.17) \quad K_\phi = \sqrt{\gamma} \partial_x, \quad K_\pi = \text{Identity}.$$

Based on this expression, we separate the equations of motion (1.11), into “material relations”

$$(1.18) \quad f_\phi(x) = K_\phi \phi(x) = \sqrt{\gamma} \partial_x \phi(x)$$

$$(1.19) \quad f_\pi(x) = K_\pi \pi(x) = \pi(x),$$

and dynamical equations,

$$(1.20) \quad \begin{aligned} \partial_t \phi(x, t) &= K_\pi^\dagger f_\pi(x, t) \\ \partial_t \pi(x, t) &= -K_\phi^\dagger f_\phi(x, t) + f(x, t). \end{aligned}$$

To obtain a first order TDD system leading to (1.4), we follow [2, 3] and replace the identity (1.19) with the “material relation,”

$$(1.21) \quad f_\pi(x, t) + \int_0^\infty \chi(x, \tau) f_\pi(x, t - \tau) d\tau = \pi(x, t),$$

but maintain (1.18) and the equations of motion (1.20), i.e.,

$$(1.22) \quad \begin{aligned} \partial_t \phi(x, t) &= f_\pi(x, t) \\ \partial_t \pi(x, t) &= \gamma \partial_x^2 \phi(x, t) + f(x, t). \end{aligned}$$

(One could also introduce dispersion in the relation between ϕ and f_ϕ , or even dispersion mixing f_ϕ and f_π , but for simplicity we consider here only TDD in the momentum.) The key requirements on the *susceptibility* χ are

- (1) Causality, manifest in (1.21) in that the integral on the l.h.s. depends only on the past.
- (2) Power dissipation, which is the requirement that for every x

$$(1.23) \quad \int_{-\infty}^\infty g(t) \int_0^\infty \chi(x, \tau) \partial_t g(t - \tau) d\tau dt \geq 0$$

for an arbitrary function $g(t)$, say compactly supported.

The significance of the power dissipation condition is as follows. We suppose that the internal energy of the TDD system at time t is given by

$$(1.24) \quad H_0 = \frac{1}{2} \int_{\mathbb{R}} (f_\pi(x, t)^2 + \gamma \partial_x \phi(x, t)^2) dx,$$

with $f_\pi(x, t)$ as in (1.21). Then

$$(1.25) \quad \begin{aligned} &\frac{d}{dt} H_0(t) \\ &= \int_{\mathbb{R}} (\gamma \partial_x \phi(x, t) \partial_t \partial_x \phi(x, t) + f_\pi(x, t) \partial_t f_\pi(x, t)) dx \\ &= \int_{\mathbb{R}} \left(\gamma \partial_x \phi(x, t) \partial_x f_\pi(x, t) + f_\pi(x, t) \partial_t \pi(x, t) \right. \\ &\quad \left. - f_\pi(x, t) \int_0^\infty \chi(x, \tau) \partial_t f_\pi(x, t - \tau) d\tau \right) dx \\ &= \int_{\mathbb{R}} \left(\gamma \partial_x \phi(x, t) \partial_x f_\pi(x, t) + \gamma f_\pi(x, t) \partial_x^2 \phi(x, t) \right. \\ &\quad \left. + f_\pi(x, t) f(x, t) - f_\pi(x, t) \int_0^\infty \chi(x, \tau) \partial_t f_\pi(x, t - \tau) d\tau \right) dx \\ &= \int_{\mathbb{R}} f(x, t) \partial_t \phi(x, t) dx - \int_{\mathbb{R}} \left[\int_0^\infty \chi(x, \tau) \partial_t^2 \phi(x, t - \tau) d\tau \right] \partial_t \phi(x, t) dx, \end{aligned}$$

where in going from the third to the final line we have used integration by parts and the assumption that $\partial_x \phi$ and f_π vanish at spatial infinity. The first term on the r.h.s. is the rate of work done on the system by the external force f . Similarly, we interpret the second term as the rate of work done by the dissipative force, $-\int \chi(\tau) \partial_t^2 \phi(x, t - \tau) d\tau$. Thus, power dissipation amounts to the requirement that, for any trajectory, the *total work done by the dissipative force is negative*.

Let us note that the right hand side of (1.23) is equal to

$$(1.26) \quad \chi(x, 0) \int_{-\infty}^{\infty} g(t)^2 dt + \int_{-\infty}^{\infty} g(t) \int_0^{\infty} \partial_\tau \chi(x, \tau) g(t - \tau) d\tau dt \\ = \int_{\mathbb{R}^2} \{ \chi(x, 0) \delta(t_1 - t_2) + \frac{1}{2} [\partial_\tau \chi](x, |t_1 - t_2|) \} g(t_1) g(t_2) d^2 t,$$

which follows from integration by parts after noting that $\partial_t g(t - \tau) = -\partial_\tau g(t - \tau)$. Here $\delta(t)$ is the Dirac delta function. Thus the power dissipation condition is equivalent to the statement that for every x the (generalized) function

$$(1.27) \quad D_\chi(x, t) = \chi(x, 0) \delta(t) + \frac{1}{2} [\partial_\tau \chi](x, |t|)$$

is *positive definite* in the sense of the classical Bochner's Theorem, see [7, Theorem IX.9]. Thus, by Bochner's Theorem, the time Fourier transform of D is a nonnegative measure,

$$(1.28) \quad \widehat{D}_\chi(x, \omega) = \chi(x, 0) + \frac{1}{2} \int_{-\infty}^{\infty} [\partial_\tau \chi](x, |t|) e^{i\omega t} dt \\ = \chi(x, 0) + \int_0^{\infty} \partial_\tau \chi(x, \tau) \cos(\omega \tau) d\tau \\ = \omega \int_0^{\infty} \chi(x, \tau) \sin(\omega \tau) d\tau = \omega \operatorname{Im} \widehat{\chi}(x, \omega) \geq 0.$$

Since χ is real, $\widehat{D}_\chi(x, \omega)$ is symmetric under $\omega \leftrightarrow -\omega$, and indeed Bochner's Theorem shows that a symmetric measure of suitably bounded growth is nonnegative if and only if it is the Fourier transform of a real positive definite distribution.

The simplest physically relevant example of a susceptibility satisfying the power dissipation condition is obtained with $\chi(x, \tau) = \alpha > 0$, a positive constant. Then

$$(1.29) \quad f_\pi(x, t) + \alpha \int_0^{\infty} f_\pi(x, t - \tau) d\tau = \pi(x, t).$$

Using the equation of motion $f_\pi = \partial_t \phi$, we find that

$$(1.30) \quad \pi(x, t) = \partial_t \phi(x, t) + \alpha \int_0^{\infty} \partial_t \phi(x, t - \tau) d\tau = \partial_t \phi(x, t) + \alpha \phi(x, t),$$

where we have applied the boundary condition $\lim_{t \rightarrow -\infty} \phi(x, t) = 0$. Combined with the equation of motion $\partial_t \pi = \gamma \partial_x^2 \phi + f$ we obtain

$$(1.31) \quad \partial_t^2 \phi(x, t) + \alpha \partial_t \phi(x, t) - \gamma \partial_x^2 \phi(x, t) = f(x, t),$$

which is the dynamical equation for a driven damped string, with damping force per unit length $-\alpha \partial_t \phi(x, t)$. Note that α is dimensionally an inverse time and $1/\alpha$ is the characteristic time for the damping of oscillations.

A more realistic model for damping is obtained by supposing $\chi(x, \tau)$ to be a nontrivial function of τ as in (1.21). To allow for damping restricted to only a part

of the string, we suppose that χ depends on x as well. For instance, we could take the Debye susceptibility

$$(1.32) \quad \chi(x, \tau) = \alpha(x)e^{-\nu(x)\tau},$$

with $\alpha(x) \geq 0$ and $\nu(x) \geq 0$ nonnegative functions of x . This results in the following integral-differential equation for ϕ

$$(1.33) \quad \partial_t^2 \phi(x, t) + \alpha(x) \int_0^\infty e^{-\nu(x)\tau} \partial_t^2 \phi(x, t - \tau) d\tau - \gamma \partial_x^2 \phi(x, t) = f(x, t),$$

or after integration by parts

$$(1.34) \quad \begin{aligned} \partial_t^2 \phi(x, t) + \alpha(x) \partial_t \phi(x, t) - \alpha(x) \nu(x) \phi(x, t) \\ + \alpha(x) \nu(x)^2 \int_0^\infty e^{-\nu(x)\tau} \phi(x, t - \tau) d\tau - \gamma \partial_x^2 \phi(x, t) = f(x, t). \end{aligned}$$

Then

$$(1.35) \quad \widehat{D}_\chi(x, \omega) = \alpha(x) \int_0^\infty e^{-\nu(x)\tau} \omega \sin(\omega\tau) d\tau = \alpha(x) \frac{\omega^2}{\nu(x)^2 + \omega^2} \geq 0,$$

so the Debye susceptibility satisfies the power dissipation condition.

The results of [2, 3] show that under the above conditions it is possible to find a coupling function $\varsigma(x, s)$, $s \in \mathbb{R}$, such that solutions to the TDD equations are generated by solutions to the following extended system

$$(1.36) \quad \begin{aligned} \partial_t \phi(x, t) &= f_\pi(x, t) \\ \partial_t \pi(x, t) &= \gamma \partial_x^2 \phi(x, t) + f(x, t) \\ \partial_t \psi(x, s, t) &= \theta(x, s, t) \\ \partial_t \theta(x, s, t) &= \partial_s^2 \psi(x, s, t) + \varsigma(x, s) f_\pi(x, t), \end{aligned}$$

with

$$(1.37) \quad f_\pi(x, t) = \pi(x, t) - \int_{-\infty}^\infty \varsigma(x, s) \psi(x, s, t) ds.$$

That is, given a solution $(\phi, \pi, \psi, \theta)$ to (1.36), at rest at $t = -\infty$ and with f_π given by (1.37), the first two coordinates (ϕ, π) obey (1.22) with f_π given by (1.21), and conversely any solution to (1.21)–(1.22) with the string at rest at $t = -\infty$ arises in this way. Note that for each fixed x the additional variables θ, ψ may be interpreted as describing the oscillations of a “hidden string” with coordinate s , displacement $\psi(x, s, t)$, momentum $\theta(x, s, t)$ and driven by external force $\varsigma(x, s) f_\pi(x, t)$. Thus (1.36) is precisely the system of coupled strings described above and illustrated in Figure 1.

The extended system, consisting of the physical and hidden strings is Hamiltonian with symplectic form

$$(1.38) \quad \begin{aligned} \mathcal{J}(\phi_1, \pi_1, \psi_1, \theta_1; \phi_2, \pi_2, \psi_2, \theta_2) &= \int_{\mathbb{R}} \{\phi_1(x) \pi_2(x) - \phi_2(x) \pi_1(x)\} dx \\ &+ \int_{\mathbb{R}^2} \{\psi_1(x, s) \theta_2(x, s) - \psi_2(x, s) \theta_1(x, s)\} ds dx \end{aligned}$$

and Hamilton function

$$(1.39) \quad \mathcal{H}_f(\phi, \pi, \psi, \theta, t) = \frac{1}{2} \int_{\mathbb{R}} \{f_\pi(x)^2 + \gamma(\partial_x \phi(x))^2\} dx - \int_{\mathbb{R}} f(x, t)\phi(x) dx \\ + \frac{1}{2} \int_{\mathbb{R}^2} \{\theta(x, s)^2 + (\partial_s \psi(x, s))^2\} ds dx,$$

where f_π is given by (1.37), that is $f_\pi(x) = \pi(x) - \int \zeta(x, s)\psi(x, s) ds$.

The dynamical equation for an excitation of the hidden string at x is a driven wave equation

$$(1.40) \quad \partial_t^2 \psi(x, s, t) = \partial_s^2 \psi(x, s, t) + \zeta(x, s)f_\pi(x, t).$$

The solution to (1.40) with the hidden string at rest at $t = -\infty$ is easily written down, see (1.3):

$$(1.41) \quad \psi(x, s, t) = \frac{1}{2} \int_0^\infty \left[\int_{s-\tau}^{s+\tau} \zeta(x, r) dr \right] f_\pi(x, t - \tau) d\tau.$$

Thus

$$(1.42) \quad \int_{\mathbb{R}} \zeta(x, s)\psi(x, s, t) ds \\ = \int_0^\infty \left[\frac{1}{2} \int_{\mathbb{R}} \zeta(x, s) \int_{s-\tau}^{s+\tau} \zeta(x, r) dr ds \right] f_\pi(x, t - \tau) d\tau.$$

Comparing (1.21) and (1.37) we see that for the extension to reproduce the TDD system upon elimination of ψ and θ it is necessary and sufficient that the coupling $\zeta(x, s)$ satisfy (1.7), that is

$$(1.43) \quad \chi(x, \tau) = \frac{1}{2} \int_{\mathbb{R}} \zeta(s, x) \int_{s-\tau}^{s+\tau} \zeta(x, r) dr ds.$$

The existence of such a function, which is unique under a natural symmetry assumption, is guaranteed by the power dissipation condition [2, 3].

Indeed, if we let $\widehat{\zeta}(x, \sigma)$ denote the s -Fourier transform of ζ ,

$$(1.44) \quad \widehat{\zeta}(x, \sigma) = \int_{\mathbb{R}} \zeta(x, s)e^{i\sigma s} ds,$$

then (1.43) is equivalent to

$$(1.45) \quad \widehat{D}_\chi(x, \omega) = \widehat{\zeta}(x, \omega)\widehat{\zeta}(x, -\omega),$$

so it suffices to take [2]

$$(1.46) \quad \widehat{\zeta}(x, \sigma) = \sqrt{2\widehat{D}_\chi(x, \sigma)}.$$

Furthermore, this choice of $\widehat{\zeta}(x, \sigma)$ is unique if we ask further that $\widehat{\zeta}(x, \sigma) \geq 0$ and that $\sigma \mapsto \widehat{\zeta}(x, \sigma)$ be symmetric. Then,

$$(1.47) \quad \zeta(x, s) = \frac{1}{2\pi} \int_{\mathbb{R}} \sqrt{2\widehat{D}_\chi(x, \sigma)} e^{-i\sigma s} d\sigma = \frac{1}{2\pi} \int_{\mathbb{R}} \sqrt{2\widehat{D}_\chi(x, \sigma)} \cos(\sigma s) d\sigma,$$

is symmetric and positive definite.

For example, in view of (1.28), (1.35), and (1.47), the following coupling produces the Debye susceptibility,

$$(1.48) \quad \zeta(x, s) = \sqrt{2\alpha(x)} \partial_s \Psi(\nu(x)s).$$

where

$$(1.49) \quad \begin{aligned} \Psi(s) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\omega} \sqrt{\frac{\omega^2}{1+\omega^2}} e^{i\omega s} d\omega = \frac{s}{|s|} \frac{1}{\pi} \int_0^1 \frac{1}{\sqrt{1-u^2}} e^{-u|s|} du \\ &= \frac{s}{|s|} \frac{1}{\pi} \int_0^{\pi/2} e^{-|s| \sin(\phi)} d\phi. \end{aligned}$$

That is,

$$(1.50) \quad \zeta(x, s) = \sqrt{2\alpha(x)} \left(\delta(s) - \frac{\nu(x)}{\pi} \int_0^{\pi/2} \sin(\phi) e^{-\nu(x)|s| \sin(\phi)} d\phi \right).$$

2. Local energy and momentum conservation in the extended system

We interpret the Hamiltonian \mathcal{H}_0 with $f \equiv 0$ as the internal energy of the damped string system consisting of the coupled physical and hidden strings. We have conservation of energy in the extended system, in the form

$$(2.1) \quad \frac{d}{dt} \mathcal{H}_0 = \int_{\mathbb{R}} f(x, t) \partial_t \phi(x, t) dx,$$

i.e., the rate of change of \mathcal{H}_0 is the rate of work done on the system by the external force.

A significant advantage of working with the extend system is a transparent interpretation of the energy of the dissipative string as a sum of contributions from the physical and hidden strings. That is, it is natural to break the \mathcal{H}_0 into two pieces

$$(2.2) \quad \mathcal{H}_0(\phi, \pi, \psi, \theta, t) = H_0(\phi, \pi, \psi) + H_{\text{hs}}(\psi, \theta),$$

the energy of the physical string and hidden strings respectively,

$$(2.3) \quad H_0(\phi, \pi, \psi) = \frac{1}{2} \int_{\mathbb{R}} \{f_{\pi}(x)^2 + \gamma(\partial_x \phi(x))^2\} dx$$

$$(2.4) \quad H_{\text{hs}}(\psi, \theta) = \frac{1}{2} \int_{\mathbb{R}^2} \{\theta(x, s)^2 + (\partial_s \psi(x, s))^2\} ds dx,$$

with f_{π} given by (1.37).

The internal energy \mathcal{H}_0 can be written as the integral of a local energy density

$$(2.5) \quad \mathcal{E}(x, t) = E_0(x, t) + E_{\text{hs}}(x, t)$$

with

$$(2.6) \quad \begin{aligned} E_0(x, t) &= \frac{1}{2} \{(\partial_t \phi(x, t))^2 + \gamma(\partial_x \phi(x, t))^2\} \\ E_{\text{hs}}(x, t) &= \frac{1}{2} \int_{\mathbb{R}} \{(\partial_t \psi(x, s, t))^2 + (\partial_s \psi(x, s, t))^2\} ds. \end{aligned}$$

The energy conservation law (2.1) has the following local expression

$$(2.7) \quad \partial_t \mathcal{E}(x, t) + \partial_x J(x, t) = f(x, t) \partial_t \phi(x, t)$$

with the energy current

$$(2.8) \quad J(x, t) = -\gamma f_{\pi}(x, t) \partial_x \phi(x, t) = -\gamma \partial_t \phi(x, t) \partial_x \phi(x, t).$$

It is interesting to compute the time derivatives of E_0 and E_{hs} alone:

$$(2.9) \quad \partial_t E_0(x, t) + \partial_x J(x, t) = f(x, t) \partial_t \phi(x, t) - \partial_t \Delta(x, t) \partial_t \phi(x, t)$$

$$(2.10) \quad \partial_t E_{\text{hs}}(x, t) = \partial_t \Delta(x, t) \partial_t \phi(x, t)$$

with

$$(2.11) \quad \begin{aligned} \Delta(x, t) = \pi(x, t) - f_\pi(x, t) &= \int_{\mathbb{R}} \varsigma(x, s) \psi(x, s, t) \, ds \\ &= \int_0^\infty \chi(x, \tau) \partial_t \phi(x, t - \tau) \, d\tau, \end{aligned}$$

where we have used (1.42). The first of these equations (2.9) is simply the local version of the energy law for the TDD string (1.25). From the second (2.10), we see that the energy of the hidden strings, which is the energy lost to dissipation up to time t , is

$$(2.12) \quad \begin{aligned} E_{\text{hs}}(x, t) &= \int_{-\infty}^t \partial_{t'} \phi(x, t') \int_0^\infty \chi(x, \tau) \partial_{t'}^2 \phi(x, t' - \tau) \, d\tau \, dt' \\ &= \int_{-\infty}^t \int_{-\infty}^t D_\chi(x, t_1 - t_2) \partial_{t_1} \phi(x, t_1) \partial_{t_2} \phi(x, t_2) \, dt_1 \, dt_2. \end{aligned}$$

If the susceptibility χ — and hence the coupling ς — is independent of x , then the extended system is invariant under spatial translations. Associated to this symmetry is a local conservation law

$$(2.13) \quad \partial_t p(x, t) + \partial_x T(x, t) = -f(x, t) \partial_x \phi(x, t),$$

for the wave momentum density

$$(2.14) \quad \begin{aligned} p(x, t) &= -\pi(x, t) \partial_x \phi(x, t) - \int_{\mathbb{R}} \theta(x, s, t) \partial_x \psi(x, s, t) \, ds \\ &= -\{\partial_t \phi(x, t) + \Delta(x, t)\} \partial_x \phi(x, t) - \int_{\mathbb{R}} \partial_t \psi(x, s, t) \partial_x \psi(x, s, t) \, ds, \end{aligned}$$

with wave momentum flux, called *stress*,

$$(2.15) \quad \begin{aligned} T(x, t) &= E_0(x, t) + \Delta(x, t) \partial_t \phi(x, t) \\ &\quad + \int \{(\partial_t \psi(x, s, t))^2 - (\partial_s \psi(x, s, t))^2\} \, ds. \end{aligned}$$

When the driving force vanishes, $f = 0$, the total wave momentum

$$(2.16) \quad P = \int_{\mathbb{R}} p(x, t) \, dx$$

is a conserved quantity.

The wave momentum density $p = p_0 + p_{\text{hs}}$ is again a sum of contributions

$$(2.17) \quad p_0(x, t) = -\{\partial_t \phi(x, t) + \Delta(x, t)\} \partial_x \phi(x, t)$$

$$(2.18) \quad p_{\text{hs}}(x, t) = - \int_{\mathbb{R}} \partial_t \psi(x, s, t) \partial_x \psi(x, s, t) \, ds$$

from the physical and hidden strings. Likewise we separate the stress

$$(2.19) \quad T(x, t) = T_0(x, t) + T_{\text{hs}}(x, t)$$

into two pieces,

$$(2.20) \quad T_0(x, t) = E_0(x, t) + \Delta(x, t) \partial_t \phi(x, t)$$

$$(2.21) \quad T_{\text{hs}}(x, t) = \frac{1}{2} \int \{(\partial_t \psi(x, s, t))^2 - (\partial_s \psi(x, s, t))^2\} \, ds.$$

Then

$$(2.22) \quad \partial_t p_0(x, t) + \partial_x T_0(x, t) = -f(x, t) \partial_x \phi(x, t) + \partial_x \Delta(x, t) \partial_t \phi(x, t)$$

$$(2.23) \quad \partial_t p_{\text{hs}}(x, t) + \partial_x T_{\text{hs}}(x, t) = -\partial_x \Delta(x, t) \partial_t \phi(x, t).$$

When we study eigenfunctions below, it will be convenient to work with complex valued solutions. In the above expressions, terms which are quadratic in the field variables should be modified in the complex case by the replacement $ab \rightarrow \text{Re } \bar{a}b$. That is

$$(2.24) \quad \begin{aligned} E_0(x, t) &= \frac{1}{2} \{ |\partial_t \phi(x, t)|^2 + \gamma |\partial_x \phi(x, t)|^2 \} \\ E_{\text{hs}}(x, t) &= \frac{1}{2} \int_{\mathbb{R}} \{ |\partial_t \psi(x, s, t)|^2 + |\partial_s \psi(x, s, t)|^2 \} ds \\ J(x, t) &= -\gamma \text{Re } \overline{\partial_t \phi(x, t)} \partial_x \phi(x, t) \\ p_0(x, t) &= -\text{Re } \overline{\partial_t \phi(x, t)} \partial_x \phi(x, t) - \text{Re } \overline{\Delta(x, t)} \partial_x \phi(x, t), \end{aligned}$$

etc.

3. The eigenfunction equation

It is useful and interesting to study steady state solutions to the extended system (1.36), for example solutions which are periodic in time $e^{-i\omega t}(\phi_\omega(x), \pi_\omega(x), \psi_\omega(x, s), \theta_\omega(x, s))$. We refer to the spatial component $\Phi_\omega(x, s) = (\phi_\omega(x), \pi_\omega(x), \psi_\omega(x, s), \theta_\omega(x, s))$ of such a time periodic solution as an eigenfunction for the linear system (1.36) with eigenvalue ω . Thus an eigenfunction satisfies

$$(3.1) \quad \begin{aligned} -i\omega \phi_\omega(x) &= f_\pi(x) \\ -i\omega \pi_\omega(x) &= \gamma \partial_x^2 \phi_\omega(x) \\ -i\omega \psi_\omega(x, s) &= \theta_\omega(x, s) \\ -i\omega \theta_\omega(x, s) &= \partial_s^2 \psi_\omega(x, s) + \varsigma(x, s) f_\pi(x), \end{aligned}$$

with

$$(3.2) \quad f_\pi(x) + \int_{\mathbb{R}} \varsigma(x, s) \psi_\omega(x, s) ds - \pi_\omega(x) = 0.$$

We see that the displacement of the hidden string at position x satisfies

$$(3.3) \quad -\partial_s^2 \psi_\omega(x, s) - \omega^2 \psi_\omega(x, s) = -i\omega \varsigma(x, s) \phi_\omega(x),$$

so

$$(3.4) \quad \begin{aligned} \psi_\omega(x, s) &= a(x) \cos(\omega s) + b(x) \sin(\omega s) - \frac{i\omega \phi_\omega(x)}{2\pi} \text{P. V.} \int_{-\infty}^{\infty} e^{-i\sigma s} \frac{1}{\sigma^2 - \omega^2} \widehat{\varsigma}(x, \sigma) d\sigma \\ &= a(x) \cos(\omega s) + b(x) \sin(\omega s) + \frac{i\phi_\omega(x)}{2} \int_{-\infty}^{\infty} \sin(\omega |s' - s|) \varsigma(x, s') ds', \end{aligned}$$

with $a(x)$ and $b(x)$ undetermined functions of x . Here, P. V. denotes the “principle value” integral,

$$\begin{aligned}
(3.5) \quad \text{P. V.} \int_{-\infty}^{\infty} e^{-i\sigma s} \frac{1}{\sigma^2 - \omega^2} \widehat{\zeta}(x, \sigma) \, d\sigma \\
&= \lim_{\delta \downarrow 0} \int_{\{\sigma: |\sigma^2 - \omega^2| \geq \delta\}} e^{-i\sigma s} \frac{1}{\sigma^2 - \omega^2} \widehat{\zeta}(x, \sigma) \, d\sigma \\
&= \lim_{\delta \downarrow 0} \frac{1}{2} \int_{\mathbb{R}} e^{-i\sigma s} \left[\frac{1}{\sigma^2 - (\omega + i\delta)^2} + \frac{1}{\sigma^2 - (\omega - i\delta)^2} \right] \widehat{\zeta}(x, \sigma) \, d\sigma.
\end{aligned}$$

By (3.4), we see that

$$\begin{aligned}
(3.6) \quad \int_{\mathbb{R}} \varsigma(x, s) \psi_{\omega}(x, s) \, ds &= a(x) \widehat{\zeta}(x, \omega) \\
&\quad - \frac{\phi_{\omega}(x)}{2\pi} \text{P. V.} \int_{-\infty}^{\infty} \frac{i\omega}{\sigma^2 - \omega^2} \widehat{\zeta}(x, \sigma) \widehat{\zeta}(x, -\sigma) \, d\sigma.
\end{aligned}$$

Recalling (1.45), that $\widehat{\zeta}(x, \sigma) \widehat{\zeta}(x, -\sigma) = \widehat{D}_{\chi}(x, \sigma) = 2\sigma \text{Im} \widehat{\chi}(x, \sigma)$, we see that the final term on the r.h.s. can be expressed

$$\begin{aligned}
(3.7) \quad - \frac{\phi_{\omega}(x)}{2\pi} \text{P. V.} \int_{-\infty}^{\infty} \frac{2i\omega}{\sigma^2 - \omega^2} \sigma \text{Im} \widehat{\chi}(x, \sigma) \, d\sigma \\
&= - \frac{i\phi_{\omega}(x)}{2\pi} \text{P. V.} \int_{-\infty}^{\infty} \left\{ \frac{1}{\sigma - \omega} + \frac{1}{-\sigma - \omega} \right\} \sigma \text{Im} \widehat{\chi}(x, \sigma) \, d\sigma \\
&= -i\omega \text{Re} \widehat{\chi}(x, \omega) \phi_{\omega}(x),
\end{aligned}$$

where we have used the symmetry $\sigma \text{Im} \widehat{\chi}(x, \sigma) = -\sigma \text{Im} \widehat{\chi}(x, -\sigma)$ and the “Kramers-Krönig relation”

$$(3.8) \quad \omega \text{Re} \widehat{\chi}(x, \omega) = \text{P. V.} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\sigma - \omega} \sigma \text{Im} \widehat{\chi}(x, \sigma) \, d\sigma.$$

Plugging this result into (3.2) and combining the first two equations of (3.1) we find that the string displacement ϕ solves the following elliptic problem

$$(3.9) \quad \gamma \partial_x^2 \phi_{\omega}(x) = -\omega^2 (1 + \text{Re} \widehat{\chi}(x, \omega)) \phi_{\omega}(x) - i\omega a(x) \widehat{\zeta}(x, \omega).$$

In analyzing (3.9) we should distinguish two cases:

- (1) $\varsigma(x, \omega) = 0$ for all x .
- (2) $\varsigma(x, \omega)$ nonzero for x in an open set.

In the first case, which is nongeneric, the medium described by the hidden strings is not absorbing at the given frequency ω , that is $\widehat{\chi}(x, \omega)$ is real for every x . Thus (3.9) is a very strong restriction on ϕ_{ω} , namely that it should satisfy the Schrödinger equation

$$(3.10) \quad (-\gamma \partial_x^2 + V(x)) \phi_{\omega}(x) = \lambda \phi_{\omega}(x),$$

with potential $V(x) = -\omega^2 \widehat{\chi}(x, \omega)$ and spectral parameter $\lambda = \omega^2$. In the second case, the physical string displacement may be decomposed as follows

$$(3.11) \quad \phi_{\omega}(x) = \phi_{\omega}^{(1)}(x) + \phi_{\omega}^{(2)}(x),$$

where $\phi_{\omega}^{(1)}(x)$ is any solution to the nondissipative eigenfunction equation

$$(3.12) \quad \gamma \partial_x^2 \phi_{\omega}^{(1)}(x) = -\omega^2 (1 + \text{Re} \widehat{\chi}(x, \omega)) \phi_{\omega}^{(1)}(x),$$

and $\phi_\omega^{(2)}$ is an arbitrary function with support in $\{x : \widehat{\zeta}(x, \omega) \neq 0\}$. Indeed, we need only choose a to be

$$(3.13) \quad a(x) = i \frac{\gamma \partial_x^2 \phi_\omega^{(2)}(x) + \omega^2 (1 + \operatorname{Re} \widehat{\chi}(x, \omega)) \phi_\omega^{(2)}(x)}{\omega \widehat{\zeta}(x, \omega)}.$$

Thus, the physical string displacement for an eigenmode $\phi_\omega(x)$ may be chosen arbitrarily within that part of the string which is absorbing at the given frequency ω . We want to emphasize the significance of this fact, since the formal eigenvalue problem

$$(3.14) \quad \gamma \partial_x^2 \phi_\omega(x) = -\omega^2 (1 + \widehat{\chi}(x, \omega)) \phi_\omega(x)$$

does not allow for such arbitrariness in the choice of $\phi_\omega(x)$. The resolution to this apparent contradiction lies in recognizing that a TDD system is an open part of a larger conservative Hamiltonian system. It is only for the extended Hamiltonian system that the eigenmodes Φ_ω are unambiguously defined with $e^{i\omega t} \Phi_\omega$ a stationary solution to the canonical Hamiltonian evolution equation. The “legitimate” eigenmodes for the original TDD string consist of projections ϕ_ω of the eigenmodes Φ_ω onto the subspace of the physical string. Thus a TDD string, being an open system has as many eigenmodes, i.e stationary solutions, as its the minimal conservative extension introduced and described in [1–3]. In particular, a finite-dimensional TDD system typically has infinitely many stationary solutions and, hence, infinitely many eigenmodes. Another view on the construction of eigenmodes follows.

The eigenfunctions written down above, involving as they do arbitrary excitations of the hidden strings, may not be relevant to the dynamics (1.36) with the external force acting on the physical string. Indeed, note that the coefficient $b(x)$ does not appear in the effective equation (3.9) for ϕ_ω . To see which eigenfunctions appear in the expansion of a general solution to (1.36) with a compactly supported external force, it is convenient to introduce the Fourier-Laplace transform

$$(3.15) \quad \widetilde{h}(\zeta) = \int_{\mathbb{R}} e^{i\zeta t} h(t) dt,$$

defined for

- (1) $\zeta \in \mathbb{C}$ if $h \rightarrow 0$ super exponentially fast as $|t| \rightarrow \infty$, for instance if h is compactly supported in time.
- (2) $\operatorname{Im} \zeta > 0$ if $h \rightarrow 0$ super exponentially fast as $t \rightarrow -\infty$.
- (3) $\operatorname{Im} \zeta < 0$ if $h \rightarrow 0$ super exponentially fast as $t \rightarrow \infty$.

Taking the Fourier–Laplace transform of (1.36) results in the following equations

$$(3.16) \quad \begin{aligned} -i\zeta \widetilde{\phi}(x, \zeta) &= \widetilde{f}_\pi(x, \zeta) \\ -i\zeta \widetilde{\pi}(x, \zeta) &= \gamma \partial_x^2 \widetilde{\phi}(x, \zeta) + \widetilde{f}(x, \zeta) \\ -i\zeta \widetilde{\psi}(x, s, \zeta) &= \widetilde{\theta}(x, s, \zeta) \\ -i\zeta \widetilde{\theta}(x, s, \zeta) &= \partial_s^2 \widetilde{\psi}(x, s, \zeta) + \varsigma(x, s) \widetilde{f}_\pi(x, \zeta), \end{aligned}$$

with

$$(3.17) \quad \widetilde{f}_\pi(x, \zeta) = \widetilde{\pi}(x, \zeta) - \int_{-\infty}^{\infty} \varsigma(x, s) \widetilde{\psi}(x, s, \zeta) ds.$$

Due to our consideration of solutions with the strings at rest at $t = -\infty$, we expect the quantities in (3.16) to be well defined only for $\operatorname{Im} \zeta > 0$. However,

if the driving force is compactly supported in time then $\tilde{f}(x, \zeta)$ is defined for all $\zeta \in \mathbb{C}$, and we may extend $\tilde{\phi}(x, \zeta)$ to $\zeta \in \mathbb{C} \setminus \mathbb{R}$ by solving (3.16). Following the eigenfunction analysis, we obtain, with $\varepsilon = \text{sign of } \text{Im } \zeta$,

$$(3.18) \quad \begin{aligned} \tilde{\psi}(x, s, \zeta) &= -\frac{i\zeta\tilde{\phi}(x, \zeta)}{2\pi} \int_{-\infty}^{\infty} e^{-i\sigma s} \frac{1}{\sigma^2 - \zeta^2} \widehat{\zeta}(x, \sigma) d\sigma \\ &= \varepsilon \frac{\tilde{\phi}(x, \zeta)}{2} \int_{-\infty}^{\infty} e^{i\varepsilon\zeta|s'-s|} \zeta(x, s') ds' \end{aligned}$$

$$(3.19) \quad \gamma \partial_x^2 \tilde{\phi}(x, \zeta) = -\zeta^2(1 + \widehat{\chi}(x, \varepsilon\zeta)) \tilde{\phi}(x, \zeta) + \tilde{f}(x, \zeta),$$

where $\widehat{\chi}(x, \zeta)$ is defined for $\text{Im } \zeta > 0$ as

$$(3.20) \quad \widehat{\chi}(x, \zeta) = \int_0^{\infty} e^{i\tau\zeta} \chi(x, \tau) d\tau.$$

For a function h vanishing at $+\infty$ or $-\infty$ we have the Fourier inversion formula

$$(3.21) \quad h(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega t \mp \eta t} \tilde{h}(\omega \mp i\eta) d\omega, \quad \left(\lim_{t=\pm\infty} h(t) = 0 \right),$$

with $\eta > 0$ arbitrary. Inverting the solution to (3.16) with $\eta > 0$ or < 0 produces distinct solutions to (1.36): for $\eta > 0$ we obtain the desired *causal* solution with the strings at rest at $t = -\infty$, while for $\eta < 0$ we obtain the *anti-causal* solution with the strings at rest at $t = +\infty$.

In a certain sense we are only interested in the causal solution to (1.36) and thus to the solution to (3.16) only for ζ in the upper half-plane. However, since (3.16) involves a source term, this solution, often called the *resolvent*, is not directly expressed as a superposition of eigenfunctions. However, there is a general procedure for decomposing the solution to (3.16) into a superposition of eigenfunctions. Namely for $\omega \in \mathbb{R}$ we define

$$(3.22) \quad \begin{aligned} &(\phi_\omega(x), \pi_\omega(x), \psi_\omega(x, s), \theta_\omega(x, s)) \\ &= \lim_{\delta \rightarrow 0} (\tilde{\phi}(x, \omega + i\delta) - \tilde{\phi}(x, \omega - i\delta), \tilde{\pi}(x, \omega + i\delta) - \tilde{\pi}(x, \omega - i\delta), \\ &\quad \tilde{\psi}(x, s, \omega + i\delta) - \tilde{\psi}(x, s, \omega - i\delta), \tilde{\theta}(x, s, \omega + i\delta) - \tilde{\theta}(x, s, \omega - i\delta)). \end{aligned}$$

Since $\tilde{f}(x, \zeta)$ is continuous at each $\zeta = \omega \in \mathbb{R}$, it follows from (3.16) that $(\phi_\omega, \pi_\omega, \psi_\omega, \theta_\omega)$ is an eigenfunction for each ω . To recover the resolvent for ζ in the upper half-plane from the eigenfunctions (3.22), suppose that the external force is supported in the set $\{t : t > t_0\}$. Then

$$(3.23) \quad |\tilde{f}(x, \zeta)| \leq C e^{-\text{Im } \zeta t_0}, \quad (\text{Im } \zeta > 0).$$

Thus $\exp(-i\zeta t_0)f(x, \zeta)$ is bounded in the upper half-plane, and by analyticity we have,²

$$(3.24) \quad (\tilde{\phi}(x, \zeta), \tilde{\pi}(x, \zeta), \tilde{\psi}(x, s, \zeta), \tilde{\theta}(x, s, \zeta)) \\ = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{e^{i(\zeta - \omega)t_0}}{\omega - \zeta} (\phi_\omega(x), \pi_\omega(x), \psi_\omega(x, s), \theta_\omega(x, s)) d\omega, \quad (\text{Im } \zeta > 0)$$

expressing the solution to (3.16) as a superposition of eigenfunctions. (There is a similar formula for the advanced solution with $\text{Im } \zeta < 0$, involving an upper bound t_1 on the support of the external force.)

Now let us fix ω and suppose that we have an eigenfunction of the form (3.22). Then by (3.18) and (3.19),

$$(3.25) \quad \psi_\omega(x, s) = \frac{i\phi_\omega(x)}{2} \int_{-\infty}^{\infty} \sin(\omega|s' - s|)\zeta(x, s') ds' + \frac{g_\omega(x)}{2} \cos(\omega s)\widehat{\zeta}(x, \omega)$$

$$(3.26) \quad \gamma\partial_x^2\phi_\omega(x) = -\omega^2(1 + \text{Re } \widehat{\chi}(x, \omega))\phi_\omega(x) - i\omega^2 \text{Im } \widehat{\chi}(x, \omega)g_\omega(x),$$

with

$$(3.27) \quad g_\omega(x) = \lim_{\delta \rightarrow 0} \{\tilde{\phi}(x, \omega + i\delta) + \tilde{\phi}(x, \omega - i\delta)\}.$$

In other words, the eigenfunctions which appear in the expansion (3.24) are of the form (3.9) with $b(x) = 0$. We refer to such eigenfunctions as *spectral* eigenfunctions. By (3.24), the spectral eigenfunctions form a complete set for the description of the dynamics of the extended system (1.36). Thus, the freedom to choose g_ω in (3.26) provides us with a rich enough family of solutions to describe the dynamics of the TDD string.

Note that a solution to (3.16) satisfies the eigenvalue equation away from the spatial support of the driving force. Thus, one may try to produce an eigenfunction as the limit of a sequence of solutions to (3.16) with a sequence of driving forces supported farther and farther from the origin. When this procedure works to produce a nontrivial eigenmode, the result is of the form (3.26) with a special choice of the arbitrary function g_ω . We can obtain in this way,

(1) The *causal eigenfunctions* with $g_\omega(x) = \phi_\omega(x)$,

$$(3.28) \quad \psi_\omega(x, s) = \frac{\phi_\omega(x)}{2} \int_{-\infty}^{\infty} e^{i\omega|s' - s|}\zeta(s', x) ds'$$

$$(3.29) \quad \gamma\partial_x^2\phi_\omega(x) = -\omega^2(1 + \widehat{\chi}(x, \omega))\phi_\omega(x),$$

corresponding to a driving force in the distant past and the causal boundary condition with the strings at rest at $t = -\infty$.

(2) The *anti-causal eigenfunctions* with $g_\omega(x) = -\phi_\omega(x)$,

$$(3.30) \quad \psi_\omega(x, s) = \frac{\phi_\omega(x)}{2} \int_{-\infty}^{\infty} e^{-i\omega|s' - s|}\zeta(s', x) ds'$$

$$(3.31) \quad \gamma\partial_x^2\phi_\omega(x) = -\omega^2(1 + \overline{\widehat{\chi}(x, \omega)})\phi_\omega(x),$$

²For a general linear system the limit on the right hand side of (3.22) is defined only as a vector valued measure and the integral on the right hand side of (3.24) should be interpreted as the integral of $1/(\omega - \zeta)$ against this measure. For the systems of coupled strings considered here, however, these boundary measures are always absolutely continuous, so (3.24) holds with $(\phi_\omega, \pi_\omega, \psi_\omega, \theta_\omega)$ defined for almost every ω by (3.22).

corresponding to a driving force in the distant future and the anti-causal, or advanced, boundary condition with the strings at rest at $t = +\infty$.

4. Energy flux in an eigenfunction

The energy density \mathcal{E} in an eigenfunction, at a point x with $\widehat{\zeta}(x, \omega) \neq 0$, is typically infinite due to the contribution from the hidden strings. This result is to be expected physically as the eigenfunction represents the idealized steady propagation of a monochromatic wave through an absorbing medium with infinite heat capacity. However, due to the decoupling between the hidden strings, energy can flow only through the physical string and we expect the energy flux J to be finite. Furthermore as we shall see $\partial_x J$, which by (2.7) is formally equal to $-\partial_t \mathcal{E}$, can be nonzero at a point x , in which case the eigenfunction is a steady state in which energy is dissipated to or absorbed from the medium at x at a constant rate.

By (2.8), the energy flux for an eigenfunction is

$$(4.1) \quad J(x, t) = -\gamma \operatorname{Re} \left\{ i\omega \overline{\phi_\omega(x)} \partial_x \phi_\omega(x) \right\} = \gamma\omega \operatorname{Im} \overline{\phi_\omega(x)} \partial_x \phi_\omega(x).$$

Thus, by (3.26),

$$(4.2) \quad -\partial_x J(x) = -\omega \operatorname{Im} \overline{\phi_\omega(x)} \gamma \partial_x^2 \phi_\omega(x) = \omega^3 \operatorname{Im} \widehat{\chi}(x, \omega) \operatorname{Re} \left\{ \overline{\phi_\omega(x)} g_\omega(x) \right\}.$$

for a spectral eigenfunction, where $g_\omega(x)$ is the arbitrary function describing the excitation of the medium, as represented by the hidden strings. The energy density of the physical string

$$(4.3) \quad E_0(x) = \frac{1}{2} \{ \omega^2 |\phi_\omega(x)|^2 + \gamma |\partial_x \phi_\omega(x)|^2 \},$$

is constant in time. We have a constant rate of dissipation at each x with $-\partial_x J(x) \neq 0$,

$$(4.4) \quad \partial_t \mathcal{E}(x) = \partial_t E_{\text{hs}}(x) = \omega^3 \operatorname{Im} \widehat{\chi}(x, \omega) \operatorname{Re} \left\{ \overline{\phi_\omega(x)} g_\omega(x) \right\}.$$

Thus the eigenfunction represents a steady state situation in which energy is flowing into or out of the hidden strings at a constant rate at each point x with $\operatorname{Im} \widehat{\chi}(x, \omega) \neq 0$.

For a general spectral eigenfunction (3.26), the dissipation $\partial_t \mathcal{E}(x)$ may be positive or negative, however for a causal eigenfunction (3.29), with $g_\omega = \phi_\omega$, we have

$$(4.5) \quad \partial_t \mathcal{E}(x) = \partial_t E_{\text{hs}}(x) = \omega^3 \operatorname{Im} \widehat{\chi}(x, \omega) |\phi_\omega(x)|^2 > 0, \quad (\text{causal eigenfunction})$$

corresponding to a steady dissipation of energy from the physical string into the medium, represented by the hidden strings. Similarly, for an anti-causal eigenfunction (3.31) there is a steady flux of energy *out of the medium and into the physical string*

$$(4.6) \quad \partial_t E_{\text{hs}}(x) = -\omega^3 \operatorname{Im} \widehat{\chi}(x, \omega) |\phi_\omega(x)|^2 < 0, \quad (\text{anti-causal eigenfunction}).$$

5. Plane waves and momentum flux in an homogeneous system

Suppose that the susceptibility $\chi(x, \tau) = \chi(\tau)$ is constant for the whole range of $x \in \mathbb{R}$. Then it is interesting to look for plane wave solutions $e^{i(kx - \omega t)} (\phi_0, \pi_0, \psi_0(s), \theta_0(s))$ with ϕ_0, π_0 constants and $\psi_0(s), \theta_0(s)$ independent of x , i.e., eigenfunctions $\propto e^{ikx}$.

A first observation is that there are no causal or anti-causal eigenfunctions of this form, at least at frequencies with $\text{Im } \widehat{\chi}(\omega) > 0$. Indeed a causal solution would satisfy

$$(5.1) \quad -\gamma k^2 \phi_0 = -\omega^2 (1 + \widehat{\chi}(\omega)) \phi_0,$$

and the only nontrivial solutions to this equation are exponentially growing as $x \rightarrow \pm\infty$ unless $\widehat{\chi}(\omega)$ is real. Such evanescent waves play a role in constructing scattering states for in-homogeneous systems but are physically irrelevant in the homogeneous system.

Thus to find plane wave solutions it is necessary to look beyond causal eigenfunctions. For a plane wave spectral eigenfunction, see (3.26), we have

$$(5.2) \quad -\gamma k^2 \phi_0 = -i\omega^2 g_0 \text{Im} \widehat{\chi}(\omega) - \omega^2 (1 + \text{Re} \widehat{\chi}(\omega)) \phi_0,$$

$$(5.3) \quad \psi_0(s) = \frac{g_0}{2} \cos(\omega s) \zeta(\omega) + \frac{i\phi_0}{2} \int_{-\infty}^{\infty} \sin(\omega|s' - s|) \zeta(s') ds',$$

with g_0 an arbitrary constant describing the excitation of the hidden strings. A bounded solution results only for k real, so

$$(5.4) \quad g_0 = i\alpha\phi_0, \quad \alpha \in \mathbb{R},$$

must be a pure imaginary multiple of ϕ_0 . Furthermore, we must have

$$(5.5) \quad 1 + \text{Re} \widehat{\chi}(\omega) - \alpha \text{Im} \widehat{\chi}(\omega) \geq 0.$$

Then setting

$$(5.6) \quad \gamma k^2 = \omega^2 (1 + \text{Re} \widehat{\chi}(\omega) - \alpha \text{Im} \widehat{\chi}(\omega)),$$

we obtain a nontrivial plane wave solution.

If the medium is not dissipative at frequency ω , so $\text{Im} \widehat{\chi}(\omega) = 0$, there is a *dispersion* relation between k and ω

$$(5.7) \quad \gamma k^2 = \omega^2 (1 + \text{Re} \widehat{\chi}(\omega)), \quad (\text{Im} \widehat{\chi}(\omega) = 0).$$

At frequencies ω with nontrivial dissipation, that is $\text{Im} \widehat{\chi}(\omega) \neq 0$, there is *no relation between k and ω* . Indeed k may be chosen arbitrarily provided we take

$$(5.8) \quad \alpha = \frac{1 + \text{Re} \widehat{\chi}(\omega)}{\text{Im} \widehat{\chi}(\omega)} - \frac{\gamma k^2}{\omega^2 \text{Im} \widehat{\chi}(\omega)}.$$

By (4.1), the energy flux for a plane wave

$$(5.9) \quad J = \gamma \omega k |\phi_0|^2$$

is constant. Thus $\partial_x J = 0$ and plane wave solutions represent a steady state flow without dissipation of energy. More precisely, the dissipation of energy to the medium, as described by the hidden strings, is exactly balanced by the energy emitted from the medium. The energy density of the physical string is

$$(5.10) \quad E_0 = (\omega^2 + \gamma k^2) |\phi_0|^2$$

and the energy density of the medium, described by the hidden strings, is of course infinite.

As the system is homogeneous, we can also consider the wave momentum density and stress in a plane wave eigenfunction. By (2.17) the wave momentum density of the physical string is

$$(5.11) \quad p_0 = \omega k |\phi_0|^2 - k \text{Im} \overline{\Delta_0} \phi_0,$$

where

$$(5.12) \quad \Delta_0 = \int_{\mathbb{R}} \varsigma(s) \psi_0(s) \, ds = i\omega \phi_0 (\alpha \operatorname{Im} \widehat{\chi}(\omega) - \operatorname{Re} \widehat{\chi}(\omega)).$$

Thus

$$(5.13) \quad p_0 = \omega k \{1 + \operatorname{Re} \widehat{\chi}(\omega) - \alpha \operatorname{Im} \widehat{\chi}(\omega)\} |\phi|^2 = \frac{\gamma k^3}{\omega} |\phi_0|^2.$$

The wave momentum density of the hidden strings is infinite, since

$$(5.14) \quad p_{\text{hs}} = \omega k \int_{\mathbb{R}} |\psi_0(s)|^2 \, ds$$

by (2.18).

Of more interest is the stress, which by (2.19)–(2.20) is

$$(5.15) \quad T = E_0 + \omega \operatorname{Im} \overline{\Delta_0} \phi_0 + T_{\text{hs}},$$

where E_0 given by (5.10) and T_{hs} is formally

$$(5.16) \quad T_{\text{hs}} = \frac{1}{2} \int_{-\infty}^{\infty} \{\omega^2 |\psi_0(s)|^2 - |\partial_s \psi_0(s)|^2\} \, ds.$$

The integral on the r.h.s. of (5.16) does not converge absolutely, but we can regularize it by defining $T_{\text{hs}} = \lim_{\delta \rightarrow 0} T_{\text{hs}}(\delta)$ with

$$(5.17) \quad T_{\text{hs}}(\delta) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-\delta|s|} \{\omega^2 |\psi_0(s)|^2 - |\partial_s \psi_0(s)|^2\} \, ds.$$

To evaluate the integral in (5.17) it is useful to write

$$(5.18) \quad \begin{aligned} |\partial_s \psi_0(s)|^2 &= \frac{1}{2} \partial_s^2 |\psi_0(s)|^2 - \operatorname{Re} \overline{\psi_0(s)} \partial_s^2 \psi_0(s) \\ &= \frac{1}{2} \partial_s^2 |\psi_0(s)|^2 + \omega^2 |\psi_0(s)|^2 + \omega \varsigma(s) \operatorname{Im} \overline{\psi_0(s)} \phi_0, \end{aligned}$$

where we have used (3.3). Thus

$$(5.19) \quad \begin{aligned} T_{\text{hs}}(\delta) &= -\frac{1}{2} \int_{-\infty}^{\infty} e^{-\delta|s|} \{\frac{1}{2} \partial_s^2 |\psi_0(s)|^2 + \omega \varsigma(s) \operatorname{Im} \overline{\psi_0(s)} \phi_0\} \, ds \\ &= \frac{\delta}{2} |\psi_0(0)|^2 - \frac{\delta^2}{4} \int_{-\infty}^{\infty} e^{-\delta|s|} |\psi_0(s)|^2 \, ds \\ &\quad - \frac{1}{2} \omega \int_{-\infty}^{\infty} e^{-\delta|s|} \varsigma(s) \operatorname{Im} \overline{\psi_0(s)} \phi_0 \, ds. \end{aligned}$$

It follows that

$$(5.20) \quad T_{\text{hs}} = \lim_{\delta \rightarrow 0} T_{\text{hs}}(\delta) = -\frac{1}{2} \omega \int_{-\infty}^{\infty} \varsigma(s) \operatorname{Im} \overline{\psi_0(s)} \phi_0 \, ds = -\frac{1}{2} \omega \operatorname{Im} \overline{\Delta_0} \phi_0,$$

if, for instance, $|\varsigma(s)|$ is integrable. Thus

$$(5.21) \quad \begin{aligned} T &= E_0 + \frac{1}{2} \omega \operatorname{Im} \overline{\Delta_0} \phi_0 \\ &= \frac{1}{2} (\omega^2 (1 + \operatorname{Re} \widehat{\chi}(\omega) - \alpha \operatorname{Im} \widehat{\chi}(\omega)) + \gamma k^2) |\phi_0|^2 = \gamma k^2 |\phi_0|^2, \end{aligned}$$

which is identical to the result for the undamped string!

6. Scattering eigenfunctions for a semi-infinite medium

To close, we would like to illustrate the power of the above method by unambiguously constructing the scattering states representing an idealized description of the following experiment. Imagine that we have an long, i.e., infinite, string whose right half ($x > 0$) is subject to dispersion and dissipation with susceptibility $\chi(\tau)$. That is, the the susceptibility for the whole string is

$$(6.1) \quad \chi(x, \tau) = \begin{cases} 0 & x < 0 \\ \chi(\tau) & x > 0 \end{cases}.$$

Suppose we drive the left end of the string, at $x = -\infty$, periodically with frequency ω , sending an incoming wave toward the TDD right half of the string. After some time a steady state is reached in which a certain fraction of the incoming wave is absorbed by the dissipative half of the string and a certain fraction is reflected.

The steady state eigenfunction describing the above experiment is a causal eigenfunction since the source is at $x = -\infty$ and started in the distant past. Thus by (3.29), the string displacement satisfies

$$(6.2) \quad \gamma \partial_x^2 \phi_\omega(x) = -\omega^2 (1 + I[x > 0] \widehat{\chi}(\omega)) \phi_\omega(x),$$

where $I[x > 0] = 1$ for $x > 0$ and 0 otherwise. Indeed, this is the naive equation that one might right down for the scattering states of a TDD string. However, it is only within the context of the extended Hamiltonian system that we understand this to be only one of many possible eigenfunction equations, the choice of which is dictated by the physics under consideration, namely a source at spatial infinity in the distant past.

By (6.2), we have

$$(6.3) \quad \phi_\omega(x) = e^{ik_<x} + r e^{-ik_<x}, \quad k_< = \frac{\omega}{\sqrt{\gamma}}, \quad (x < 0)$$

on the left half of the string, up to an over all multiple which we fix to be 1 without loss. In (6.3) the term $e^{ik_<x}$ is the incoming wave and $r e^{-ik_>x}$ is the reflected wave.

To find the reflection coefficient r we need to solve for the form of the eigenfunction on the right half of the string. Again by (6.2), we have

$$(6.4) \quad \gamma \partial_x^2 \phi_\omega(x) = -\omega^2 (1 + \widehat{\chi}(\omega)) \phi_\omega(x), \quad (x > 0)$$

for $x > 0$. Furthermore, since we require the solution to be bounded this determines ϕ uniquely up to an over all multiple

$$(6.5) \quad \phi_\omega(x) = v e^{ik_>x},$$

where v is an, as yet, undetermined transmission coefficient and

$$(6.6) \quad k_> = \sqrt[+]{\frac{\omega^2}{\gamma} (1 + \widehat{\chi}(\omega))}.$$

Here $\sqrt[+]{z}$ denotes the unique square root of z in the upper half-plane.³ Since $\text{Im } k_{>} > 0$, at least if $\text{Im } \widehat{\chi}(\omega) \neq 0$, we see that $\phi_{\omega}(x)$ decays exponentially in the dissipative half of the physical string ($x > 0$). The excitation of the hidden strings, which are restricted to $x > 0$, is given by (3.28):

$$(6.7) \quad \psi_{\omega}(x, s) = \frac{v}{2} e^{ik_{>}x} \int_{-\infty}^{\infty} e^{i\omega|s-s'|} \zeta(s') ds', \quad (x > 0).$$

To determine v and r we note that the eigenfunction equation (6.2) forces ϕ_{ω} and $\partial_x \phi_{\omega}$ to be continuous functions of x , in particular at $x = 0$. Thus,

$$(6.8) \quad \lim_{x \uparrow 0} \phi_{\omega}(x) = 1 + r = v = \lim_{x \downarrow 0} \phi_{\omega}(x)$$

$$(6.9) \quad \lim_{x \uparrow 0} \partial_x \phi_{\omega}(x) = k_{<}(1 - r) = k_{>}v = \lim_{x \downarrow 0} \partial_x \phi_{\omega}(x).$$

We conclude that

$$(6.10) \quad v = \frac{2}{1 + \rho(\omega)}, \quad r = \frac{1 - \rho(\omega)}{1 + \rho(\omega)},$$

where

$$(6.11) \quad \rho(\omega) = \frac{k_{>}}{k_{<}} = \begin{cases} \sqrt[+]{1 + \widehat{\chi}(\omega)} & \omega > 0, \\ -\sqrt[+]{1 + \widehat{\chi}(\omega)} & \omega < 0. \end{cases}$$

It is useful to consider some general properties of $\rho(\omega)$ and the reflection and transmission coefficients. To begin note that since $\omega \text{Im } \widehat{\chi}(\omega) \geq 0$ we have

$$(6.12) \quad \omega \text{Re } \sqrt[+]{1 + \widehat{\chi}(\omega)} \geq 0,$$

and thus

$$(6.13) \quad \text{Re } \rho(\omega) \geq 0.$$

This implies that

$$(6.14) \quad 1 - |r|^2 = \frac{4}{1 + |\rho(\omega)|^2} \text{Re } \rho(\omega) = |v|^2 \text{Re } \rho(\omega) = |v|^2 \frac{\text{Re } k_{>}}{k_{<}} \geq 0,$$

so, in particular,

$$(6.15) \quad |r| \leq 1.$$

Equation (6.14) expresses the continuity of the energy flux $J(x)$ at $x = 0$, since by (4.1)

$$(6.16) \quad \begin{aligned} J(x) &= \gamma \omega \text{Im } \overline{\phi_{\omega}(x)} \partial_x \phi_{\omega}(x) \\ &= \gamma \omega \begin{cases} k_{<}(1 - |r|^2) & x < 0 \\ \text{Re } k_{>} |v|^2 e^{-2 \text{Im } k_{>} x} & x > 0 \end{cases} \\ &= \sqrt{\gamma} \omega^2 (1 - |r|^2) \begin{cases} 1 & x < 0, \\ e^{-2 \text{Im } k_{>} x} & x > 0. \end{cases} \end{aligned}$$

³If $\widehat{\chi}(\omega)$ is real and $(1 + \widehat{\chi}(\omega)) > 0$ then the medium transmits at frequency ω and $k_{>}$ should be determined as the limit

$$k_{>} = \lim_{\delta \downarrow 0} \sqrt[+]{\frac{(\omega + i\delta)^2}{\gamma} (1 + \widehat{\chi}(\omega + i\delta))}.$$

Furthermore, we see that the energy flux is nonnegative, representing a flow of energy from the source at $x = -\infty$, and the rate of dissipation, $-\partial_x J(x) \neq 0$, is nonzero for every $x > 0$,

$$(6.17) \quad -\partial_x J(x) = 2\sqrt{\gamma}\omega^2(1 - |r|^2) \operatorname{Im} k_{>} e^{-2\operatorname{Im} k_{>} x}, \quad (x > 0).$$

Finally, we note that the energy density of the physical string is

$$(6.18) \quad \begin{aligned} E_0(x) &= \frac{1}{2} \{ \omega^2 |\phi_\omega(x)|^2 + \gamma |\partial_x \phi_\omega(x)|^2 \} \\ &= \begin{cases} \omega^2(1 + |r|^2) & x < 0, \\ \omega^2 \frac{1}{2} (1 + |1 + \widehat{\chi}(\omega)|) |v|^2 e^{-2\operatorname{Im} k_{>} x} & x > 0, \end{cases} \\ &= \omega^2(1 + |r|^2) \begin{cases} 1 & x < 0, \\ e^{-2\operatorname{Im} k_{>} x} & x > 0, \end{cases} \end{aligned}$$

where we have noted that by (6.10)

$$(6.19) \quad 1 + |r|^2 = \frac{1 + |\rho(\omega)|^2}{2} |v|^2 = \frac{1 + |1 + \widehat{\chi}(\omega)|}{2} |v|^2.$$

In summary, the scattering eigenmode

$$(6.20) \quad \phi(x, t) = \begin{cases} e^{i\omega(x/\sqrt{\gamma}-t)} + r e^{-i\omega(x/\sqrt{\gamma}+t)} & x < 0 \\ v e^{i\omega(\rho(\omega)x/\sqrt{\gamma}-t)} & x > 0, \end{cases}$$

$$(6.21) \quad \psi(x, s, t) = \frac{v}{2} e^{i\omega(\rho(\omega)x/\sqrt{\gamma}-t)} \int_{\mathbb{R}} e^{i\omega|s-s'|} \zeta(s') ds', \quad (x > 0),$$

describes a steady state in which the incoming wave is partially reflected, with the remainder an evanescent transmitted wave that penetrates the dissipative part of the string with an exponential profile resulting in an excitation of the hidden strings accounting for dispersion and dissipation. The total rate of dissipation in the TDD portion of the string is

$$(6.22) \quad - \int_0^\infty \partial_x J(x) dx = J(0) = \sqrt{\gamma}\omega^2(1 - |r|^2).$$

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